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A LITTLEWOOD-RICHARDSON FILTRATION AT ROOTS OF 1 FOR MULTIPARAMETR DEFORMATIONS OF SKEW SCHUR MODULES

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Let R be a commutative ring, q a unit of R and P a multiplicatively antisymmetric matrix with coefficients which are integer powers of q. Denote by $SE(q,\mathrm{P})$ the multiparameter quantum matrix bialgebra associated to q and P. Slightly generalizing [H-H], we define a multiparameter deformation $L_{\lambda/\mu}\mathrm{V_P}$ of the classical skew Schur module. In case R is a field and q is not a root of 1, arguments like those given in [H-H] show that $L_{\lambda/\mu}\mathrm{V_P}$ is completely reducible, and its decomposition into irreducibles is $\sum_\nu \gamma(\lambda/\mu;\nu)L_\nu\mathrm{V_P}$, where the coefficients $\gamma(\lambda/\mu;\nu)$ are the usual Littlewood-Richardson coefficients. When R is *any* ring and q is allowed to be a root of 1, we construct a filtration of $L_{\lambda/\mu}\mathrm{V_P}$ as an $SE(q,\mathrm{P})$-comodule, such that its associated graded object is precisely $\sum_\nu \gamma(\lambda/\mu;\nu)L_\nu\mathrm{V_P}$.

1. The Ingredients

1.1 Let $N>1$ be a positive integer. Choose a unit q in a commutative ring R; fix a matrix $\mathrm{P}=(p_{ij})_{i,j=1}^N$ where the p_{ij}'s are non-zero elements of R with the property

$$p_{ij}p_{ji}=p_{ii}=1 \quad \forall i,j=1,\ldots,N.$$

Consider the free R-module $\mathrm{V_P}$ with basis $\{u_1,\ldots,u_N\}$ and define an automorphism $\beta_{q,\mathrm{P}}$ on $\mathrm{V_P}\otimes\mathrm{V_P}$ by the following rule:

$$\beta_{q,\mathrm{P}}(u_i\otimes u_j)=\begin{cases} u_i\otimes u_i & \text{if } i=j\\ qp_{ji}u_j\otimes u_i & \text{if } i<j\\ qp_{ji}u_j\otimes u_i+(1-q^2)u_i\otimes u_j & \text{if } i>j\end{cases}.$$

Then $(\mathrm{V_P},\beta_{q,\mathrm{P}})$ is a YB pair in the sense of [H-H]. Moreover it satisfies the Iwahori's quadratic equation

$$(id_{\mathrm{V_P}\otimes\mathrm{V_P}}-\beta_{q,\mathrm{P}})\circ(id_{\mathrm{V_P}\otimes\mathrm{V_P}}+q^{-2}\beta_{q,\mathrm{P}})=0,$$

as we can easily verify.

1.2 The *multiparameter quantum matrix bialgebra* $SE(q,\mathrm{P})$ [S] is the algebra generated by the N^2 elements x_{ij} $(i,j=1,\ldots,N)$ with relations (for $i<j$ and $k<m$) :

$$x_{ik}x_{im}=qp_{mk}x_{im}x_{ik} \qquad x_{ik}x_{jk}=qp_{ij}x_{jk}x_{ik} \qquad p_{mk}x_{im}x_{jk}=p_{ij}x_{jk}x_{im}$$

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$$p_{km}x_{ik}x_{jm} - p_{ij}x_{jm}x_{ik} = (q - q^{-1})x_{im}x_{jk}.$$

The coalgebra structure is given by the following comultiplication and counity :

$$\Delta(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj} \quad , \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

1.3 There is a natural $SE(q, \mathrm{P})$-comodule structure on $\mathrm{V_P}$ given by

$$u_j \mapsto \sum_i u_i \otimes x_{ij}.$$

Consider the ideal $\mathcal{B}^+_{\mathrm{P}}$ of $SE(q, \mathrm{P})$ generated by all x_{ij} with $i > j$ and put

$$SB^+(q, \mathrm{P}) = SE(q, \mathrm{P})/\mathcal{B}^+_{\mathrm{P}}.$$

The relations between the generators in $SB^+(q, \mathrm{P})$ are those given in (1.2) when we put $x_{ij} = 0$ for $i > j$. In particular x_{ii} commutes with x_{jj} for all i, j.

1.4 Henceforth the p_{ij}'s will be integer powers of q. More precisely (cf.[R]) we shall take

$$p_{ij} = q^{2(u_{ji} - u_{j-1i} - u_{ji-1} + u_{j-1i-1})},$$

where $U = (u_{ij})_{i,j=1}^{N-1}$ is an appropriate alternating integer matrix. In this way we shall be in the situation of [C-V 1-2], where in fact an integer form of the multiparameter quantum function algebra is constructed. From now on, we shall skip all indices q, P in our notations as long as no ambiguity is likely.

1.5 We now begin reviewing some results of [H-H], freely adopting the notations in there. Starting from the YB pair $(\mathrm{V}, \beta_{\mathrm{V}})$, we can construct some graded YB bialgebras. First of all the tensor algebra $T\mathrm{V} = \bigoplus_{i \geq 0} \mathrm{V}^{\otimes i} = \bigoplus_{i \geq 0} T_i\mathrm{V}$ with YB operator $T(\beta_{\mathrm{V}}) = \bigoplus_{i,j \geq 0} \beta_{\mathrm{V}}(\chi_{ij})$, where χ_{ij} is the following element of $\mathcal{S}_{i+j}$:

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \ldots & i & i+1 & i+2 & \ldots & i+j \\ j+1 & j+2 & \ldots & j+i & 1 & 2 & \ldots & j \end{pmatrix}.$$

We recall that, if $\sigma = \sigma_{i_1} \cdots \sigma_{i_r}$ is a reduced expression for an element $\sigma \in \mathcal{S}_k$, then it is well defined on $T_k\mathrm{V}$ the operator $\beta_{\mathrm{V}}(\sigma) = \beta_{\mathrm{V}}(\sigma_{i_1}) \circ \cdots \circ \beta_{\mathrm{V}}(\sigma_{i_r})$, $\beta_{\mathrm{V}}(\sigma_j)$ being the map $id_{\mathrm{V}}^{\otimes j-1} \otimes \beta_{\mathrm{V}} \otimes id_{\mathrm{V}}^{\otimes k-j-1}$. In order to describe the coproduct of $T\mathrm{V}$, for every sequence $\alpha = (\alpha_1, \ldots, \alpha_s)$ of nonnegative integers with $\sum_i \alpha_i = k$, define $\Delta^{\alpha}_{T\mathrm{V}}$ to be the composite map $T\mathrm{V} \longrightarrow T_s\mathrm{V} \longrightarrow T_{\alpha}\mathrm{V}$ of the s-th iteration of $\Delta_{T\mathrm{V}}$ and the projection onto $T_{\alpha}\mathrm{V} = \mathrm{V}^{\otimes \alpha_1} \otimes \cdots \otimes \mathrm{V}^{\otimes \alpha_s}$. Put

$$\mathcal{S}^{\alpha} = \{\sigma \in \mathcal{S}_k | \sigma(1) < \cdot\cdot < \sigma(\alpha_1), \sigma(\alpha_1 + 1) < \cdot\cdot < \sigma(\alpha_1 + \alpha_2), \ldots, \sigma(\sum_{i=1}^{s-1} \alpha_i + 1) < \cdot\cdot < \sigma(\sum_{i=1}^{s} \alpha_i)\}.$$

Then $\Delta_{T\mathrm{V}}^{\alpha} = \sum_{\sigma \in \mathcal{S}^\alpha} \beta_{\mathrm{V}}(\sigma^{-1})$.

1.6 We consider the symmetric and the exterior algebras $S\mathrm{V}$ and $\Lambda\mathrm{V}$ of the YB pair $(\mathrm{V}, \beta_{\mathrm{V}})$, which will play a key role in what follows. The algebra $S\mathrm{V}$ is generated by $u_1, \dots, u_N$ with relations

$$u_i u_j = p_{ji} q u_j u_i,$$

while $\Lambda\mathrm{V}$ is the algebra on the same generators with relations

$$u_i \wedge u_i = 0, \quad p_{ji} q u_i \wedge u_j + u_j \wedge u_i = 0 \ (i < j).$$

So for every sequence $i = (i_1, \dots, i_k)$ of elements in $[1, N]$ we have

$$u_{i_1} \wedge \cdots \wedge u_{i_k} = \begin{cases} 0 & \text{if there are repetitions in } i \\ (\prod_{r<t, \sigma(r)>\sigma(t)} -q^{-1} p_{i_{\sigma(r)} i_{\sigma(t)}}) u_{i_{\sigma(1)}} \wedge \cdots \wedge u_{i_{\sigma(r)}} & \text{if } i_1 < \cdots < i_k \text{ and } \sigma \in \mathcal{S}_k \end{cases}.$$

The R-modules $S_r\mathrm{V}$ and $\Lambda_r\mathrm{V}$ are free with bases, respectively,

$$\{u_{j_1} \cdots u_{j_r} | \ 1 \le j_1 \le \cdots \le j_r \le N\} \ , \ \{u_{j_1} \wedge \cdots \wedge u_{j_r} | \ 1 \le j_1 < \cdots < j_r \le N\}.$$

1.7 Put $\gamma_{\mathrm{V}} = -q^{-2} \beta_{\mathrm{V}}$. Then the two YB operators $\beta_{\mathrm{V}}, \gamma_{\mathrm{V}}$ satisfy conditions (4.9) and (4.10) in [H-H], that is, $(\mathrm{V}, \beta_{\mathrm{V}}, \gamma_{\mathrm{V}})$ is a YB triple. From this follows (Theorem 4.10 in [H-H]) that $S\mathrm{V}$ and $\Lambda\mathrm{V}$ are graded YB bialgebras. Moreover there exist YB operators $\varphi_{S\mathrm{V}}$, $\psi_{S\mathrm{V}}$ on $S\mathrm{V}$, and $\varphi_{\Lambda\mathrm{V}}$, $\psi_{\Lambda\mathrm{V}}$ on $\Lambda\mathrm{V}$, for which $(S\mathrm{V}, \varphi_{S\mathrm{V}}, \psi_{S\mathrm{V}})$ and $(\Lambda\mathrm{V}, \varphi_{\Lambda\mathrm{V}}, \psi_{\Lambda\mathrm{V}})$ are YB algebra triples. In particular, the operator $\varphi_{\Lambda\mathrm{V}}$ is defined by the relation $\varphi_{\Lambda\mathrm{V}} \circ (p \otimes p) = (p \otimes p) \circ T(-\beta_{\mathrm{V}})$ where p denotes the projection from $T\mathrm{V}$ onto $\Lambda\mathrm{V}$. The multiplicative structure on $\Lambda\mathrm{V}$ is given by the fusion procedure, namely, by

$$m_{T_i(\Lambda\mathrm{V})} = m_{\Lambda\mathrm{V}}^{\otimes i} \circ \varphi_{\Lambda\mathrm{V}}(\omega_i), \qquad \omega_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+3 & \cdots & 2i \\ 1 & 3 & \cdots & 2i-1 & 2 & 4 & \cdots & 2i \end{pmatrix}.$$

Finally note that $T\mathrm{V}$, $S\mathrm{V}$ and $\Lambda\mathrm{V}$ are SE-equivariant as YB bialgebras with YB algebra triples, that is, all the structure morphisms (including YB operators) are homomorphisms of SE-comodules.∎

1.8 A translation into our setting of Lemma 5.3 in [H-H] gives the following very useful equality.

Lemma *For any $k \ge 0$ and any sequence $(i_1, \dots, i_k)$ with $1 \le i_1 < \cdots < i_k \le N$ we have :*

$$\Delta_{\Lambda\mathrm{V}}^{(1,\dots,1)}(u_{i_1} \wedge \cdots \wedge u_{i_k}) = \sum_{\sigma \in \mathcal{S}_k} (\prod_{r<t, \sigma(r)>\sigma(t)} -q p_{i_{\sigma(r)} i_{\sigma(t)}}) u_{i_{\sigma(1)}} \otimes \cdots \otimes u_{i_{\sigma(k)}}.$$

In particular, for any k, $\Delta_{\Lambda\mathrm{V}} : \Lambda\mathrm{V} \longrightarrow T_k\mathrm{V}$ is a split injection. □

1.9 We are now ready to introduce our multiparameter deformations of Schur modules. In fact all definitions and results in Section 6 of [H-H], stated for the "Jimbo case", still hold in our situation.

For all but Lemma 6.12 can be deduced from formal properties of graded YB bialgebras which are also equipped with a structure of YB algebra triple. The proof of Lemma 6.12, which depends directly on the definition of β_{V}, can be easily modified for our purposes.
Given a skew partition λ/μ with $l(\lambda/\mu)=s$ and $\lambda_1=t$, denote by $d_{\lambda/\mu}(\mathrm{V})$ the *Schur map*, that is, the composite map

$$\Lambda_{\lambda/\mu}\mathrm{V}=\Lambda_{\lambda_1-\mu_1}\mathrm{V}\otimes\cdots\otimes\Lambda_{\lambda_t-\mu_t}\mathrm{V} \overset{\Delta_{\Lambda\mathrm{V}}^{(1^{\lambda_1-\mu_1})}\otimes\cdots\otimes\Delta_{\Lambda\mathrm{V}}^{(1^{\lambda_t-\mu_t})}}{\longrightarrow} T_{\lambda/\mu}\mathrm{V}=T_{\lambda_1-\mu_1}\mathrm{V}\otimes\cdots\otimes T_{\lambda_t-\mu_t}\mathrm{V}\longrightarrow$$

$$\overset{(-q^{-2}\beta_{\mathrm{V}})(\chi_{\lambda/\mu})}{\longrightarrow} T_{\tilde{\lambda}/\tilde{\mu}}\mathrm{V}=T_{\tilde{\lambda_1}-\tilde{\mu_1}}\mathrm{V}\otimes\cdots\otimes T_{\tilde{\lambda_s}-\tilde{\mu_s}}\mathrm{V}\overset{p\otimes\cdots\otimes p}{\longrightarrow} S_{\tilde{\lambda}/\tilde{\mu}}\mathrm{V}=S_{\tilde{\lambda_1}-\tilde{\mu_1}}\mathrm{V}\otimes\cdots\otimes S_{\tilde{\lambda_s}-\tilde{\mu_s}}\mathrm{V},$$

where, as usual, $\tilde{\lambda}$ denotes the dual partition of λ, and $\chi_{\lambda/\mu}$ is the permutation defined in Section 6 of [H-H]. We illustrate such a permutation by the following example :

$$\lambda=(5,4,2)\qquad \mu=(2,1)\qquad \chi_{\lambda/\mu}=\begin{pmatrix}1&2&3&4&5&6&7&8\\4&6&8&2&5&7&1&3\end{pmatrix}$$

$$\begin{array}{ccccc}\bullet&\bullet&1&2&3\\\bullet&4&5&6&\\7&8&&&\end{array}\quad\overset{\chi_{\lambda/\mu}}{\longrightarrow}\quad\begin{array}{ccccc}\bullet&\bullet&4&6&8\\\bullet&2&5&7&\\1&3&&&\end{array}\quad.$$

The image of the Schur map, denoted by $L_{\lambda/\mu}\mathrm{V}$, is the *Schur module of* V *with respect to the skew partition* λ/μ. It is an SE-comodule, with coaction induced by the following coaction on $T_k\mathrm{V}$:

$$u_{j_1}\otimes\cdots\otimes u_{j_k}\mapsto\sum_{i_1,\ldots,i_k}(u_{i_1}\otimes\cdots\otimes u_{i_k})\otimes x_{i_1j_1}\otimes\cdots\otimes x_{i_kj_k}.$$

1.10 The principal properties of $L_{\lambda/\mu}\mathrm{V}$ are summarized in the following theorem, which one proves along the lines of Theorem 6.19 and Corollary 6.20 in [H-H].

Theorem *Let λ/μ a skew partition with $l(\lambda/\mu)=s$. Then :*
(i) $L_{\lambda/\mu}\mathrm{V}$ is an R-free module, and for any $\sigma\in\mathcal{S}_N$, a free basis is the set

$$L_{\lambda/\mu}\mathrm{Y}(\sigma)=\{d_{\lambda/\mu}(\mathrm{V})(\xi_{\mathrm{S}})|\ \mathrm{S}\in St_{\lambda/\mu}\mathrm{Y}(\sigma)\}.$$

Here $St_{\lambda/\mu}\mathrm{Y}$ denotes the set of all standard tableaux in the alphabet $\mathrm{Y}(\sigma)=\{u_{\sigma(1)}<\cdots<u_{\sigma(N)}\}$, and

$$\xi_{\mathrm{S}}=\mathrm{S}(1,\mu_1+1)\wedge\cdots\wedge\mathrm{S}(1,\lambda_1)\otimes\cdots\otimes\mathrm{S}(s,\mu_s+1)\wedge\cdots\wedge\mathrm{S}(s,\lambda_s)\in\Lambda_{\lambda/\mu}\mathrm{V}.$$

(ii) Let R' be a commutative ring and let $f:R\longrightarrow R'$ be a homomorphism of commutative rings. Then we have an isomorphism of SE'-comodules

$$L_{\lambda/\mu}(R'\otimes_R\mathrm{V})\simeq R'\otimes_R L_{\lambda/\mu}\mathrm{V}\quad,\quad E'=R'\otimes_R E.$$

$\square$

As a consequence of (ii), it will not be restrictive for us to take $R = \mathbb{Z}[\mathcal{Q}, \mathcal{Q}^{-1}]$, where $\mathcal{Q}$ stands for an indeterminate.

1.11 We recall that an element of $Tab_{\lambda/\mu}\mathrm{Y}(\sigma)$, the set of all tableaux of shape λ/μ with elements in $\mathrm{Y}(\sigma)$, is said to be *row-standard* if its rows are strictly increasing, and *column-standard* if its columns are non-decreasing. A tableau is said to be *standard* if it is both row- and column-standard. Let $Row_{\lambda/\mu}\mathrm{Y}(\sigma)$ denote the set of row-standard tableaux of shape λ/μ and with elements in $\mathrm{Y}(\sigma)$. For every $\mathrm{S} \in Row_{\lambda/\mu}\mathrm{Y}(\sigma)$, the element $d_{\lambda/\mu}(\mathrm{V})(\xi_{\mathrm{S}})$ can be expressed as a linear combination of basis elements. The algorithm, call it $\mathcal{R}_\sigma$, which does this is based on a descending induction with respect to a pseudo order defined in $Tab_{\lambda/\mu}\mathrm{Y}(\sigma)$. Let S and S$'$ be elements in $Tab_{\lambda/\mu}\mathrm{Y}(\sigma)$. We say that $\mathrm{S} \leq_\sigma \mathrm{S}'$ if $\forall p, q$

$$\#\{(i,j) \in \Delta_{\lambda/\mu} |\ i \leq p,\ \mathrm{S}(i,j) \in \{u_{\sigma(1)}, \ldots, u_{\sigma(q)}\}\} \geq \#\{(i,j) \in \Delta_{\lambda/\mu} |\ i \leq p,\ \mathrm{S}'(i,j) \in \{u_{\sigma(1)}, \ldots, u_{\sigma(q)}\}\}.$$

The key steps of $\mathcal{R}_\sigma$ are the following:

1. Choose two adjacent lines in S where there is a violation of column-standardness; we are in the situation of Proposition (1.12) below, and we can use Corollary (1.13). We get certain S_i's such that $\mathrm{S}_i < \mathrm{S}$ for every i.
2. Reorder in increasing order $\mathrm{S}_i(1, \mu_1+1) \wedge \cdots \wedge \mathrm{S}_i(1, \lambda_1), \ldots, \mathrm{S}_i(s, \mu_s+1) \wedge \cdots \wedge \mathrm{S}_i(s, \lambda_s)$ for each i; this operation produces a power of q for every S_i (cf. (1.6)).
3. Apply induction to each S_i.

$\mathcal{R}_\sigma$ is also called the "straightening law with respect to the ordering $u_{\sigma(1)} < \cdots < u_{\sigma(N)}$".

1.12 Proposition *Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ be partitions with $\lambda \supset \mu$. Define $\gamma = \lambda - \mu$ and take a, b nonnegative integers with $a + b < \lambda_2 - \mu_1$. Then the image of the composite map*

$$\overline{\Box}_{(a,b)} : \Lambda_a\mathrm{V} \otimes \Lambda_{\gamma_1 - a + \gamma_2 - b}\mathrm{V} \otimes \Lambda_b\mathrm{V} \xrightarrow{1\otimes\Delta\otimes 1} \Lambda_a\mathrm{V} \otimes \Lambda_{\gamma_1 - a}\mathrm{V} \otimes \Lambda_{\gamma_2 - b}\mathrm{V} \otimes \Lambda_b\mathrm{V} \xrightarrow{m\otimes m} \Lambda_{\gamma_1}\mathrm{V} \otimes \Lambda_{\gamma_2}\mathrm{V} = \Lambda_{\lambda/\mu}\mathrm{V}$$

is contained in $Im(\Box_{\lambda/\mu})$, where $\Box_{\lambda/\mu}$ is given by

$$\sum_{\nu=0}^{\lambda_2-\mu_1} \Lambda_{\gamma_1+\gamma_2-\nu}\mathrm{V} \otimes \Lambda_\nu\mathrm{V} \xrightarrow{\Delta\otimes 1} \sum_{\nu=0}^{\lambda_2-\mu_1} \Lambda_{\gamma_1}\mathrm{V} \otimes \Lambda_{\gamma_2-\nu}\mathrm{V} \otimes \Lambda_\nu\mathrm{V} \xrightarrow{1\otimes m} \sum_{\nu=o}^{\lambda_2-\mu_1} \Lambda_{\gamma_1}\mathrm{V} \otimes \Lambda_{\gamma_2}\mathrm{V}.$$

Proof. Mimic the proof of Lemma 6.15 in [H-H]. □

1.13 Corollary *Let λ/μ be a skew partition with $l(\lambda) = s$, σ be an element of $\mathcal{S}_N$ and S be an element of $Row_{\lambda/\mu}\mathrm{Y}(\sigma) \setminus St_{\lambda/\mu}\mathrm{Y}(\sigma)$. Then there exist $\mathrm{S}_1, \ldots, \mathrm{S}_r \in Row_{\lambda/\mu}\mathrm{Y}(\sigma)$ $(r \in \mathbb{N})$ with $\mathrm{S}_i <_\sigma \mathrm{S},\ \forall i = 1, \ldots, r$ such that*

$$\xi_{\mathrm{S}} - \sum_i c_i \xi_{\mathrm{S}_i} \in Im(\Box_{\lambda/\mu}) = Ker(d_{\lambda/\mu}(\mathrm{V})),$$

for some $c_i \in \mathbb{Z}[q,q^{-1}]$. *Here :*

$$\square_{\lambda/\mu} = \sum_{i=1}^{s-1} 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \square_{\lambda^i/\mu^i} \otimes 1_{i+2} \otimes \cdots \otimes 1_s, \ \lambda^i = (\lambda_i, \lambda_{i+1}), \ \mu^i = (\mu_i, \mu_{i+1}), \ 1_j = id_{\Lambda_{\lambda_j - \mu_j}\mathrm{V}}.$$

Proof. Mimic the proof of Lemma 6.18 in [H-H]. □

1.14 We want to stress a consequence of Theorem (1.10) and of all the machinery which allows to prove it. First of all note that the subcategory of $\mathcal{YB}_R$ (cf. [H-H]) given by the YB pairs as in 1.1 is a preadditive one. Namely, let $\mathrm{P}^1 = (p^1_{ij})^n_{i,j=1}$ and $\mathrm{P}^2 = (p^2_{ij})^m_{i,j=1}$ be two multiplicatively antisymmetric matrices, and put $\mathrm{V}_{\mathrm{P}^1} =< u^1_1, \ldots, u^1_n >$, $\mathrm{V}_{\mathrm{P}^2} =< u^2_1, \ldots, u^2_m >$. We define a YB operator on $\mathrm{V}_{\mathrm{P}^1} \oplus \mathrm{V}_{\mathrm{P}^2}$ by means of the matrix $\mathrm{P} = (p_{ij})^N_{i,j=1}$, $N = n+m$, defined as follows:

$$p_{ij} = \begin{cases} p^1_{ij} & \text{for } i,j \in [1,N] \\ p^2_{ij} & \text{for } i,j \in [n+1,N] \\ 1 & \text{for } i \in [1,n], j \in [n+1,N] \text{ or } i \in [n+1,N], j \in [1,n] \end{cases}.$$

Then β_{P} is a YB operator on $\mathrm{V}_{\mathrm{P}} =< u^1_1, \ldots, u^1_n, u^2_1, \ldots, u^2_m >$. Note that V_{P} becomes in a natural way an $SE(q,\mathrm{P}^1) \otimes SE(q,\mathrm{P}^2)$-comodule.
Write for short $\mathrm{V}_i = \mathrm{V}_{\mathrm{P}^i}$, $\beta_i = \beta_{\mathrm{P}^i}$, for $i = 1,2$, and let $\mu \subset \gamma \subset \lambda$ be partitions. Following [A-B-W], define two R-modules

$$M_\gamma(\Lambda_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2)) = Im(\sum_{\mu \subseteq \sigma \subseteq \lambda, \sigma \geq \gamma} \Lambda_{\sigma/\mu}\mathrm{V}_1 \otimes \Lambda_{\lambda/\sigma}\mathrm{V}_2 \longrightarrow \Lambda_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2)),$$

$$\dot{M}_\gamma(\Lambda_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2)) = Im(\sum_{\mu \subseteq \sigma \subseteq \lambda, \sigma > \gamma} \Lambda_{\sigma/\mu}\mathrm{V}_1 \otimes \Lambda_{\lambda/\sigma}\mathrm{V}_2 \longrightarrow \Lambda_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2)),$$

where the indicated maps are obtained by tensoring the obvious maps

$$\Lambda_{\sigma_i - \mu_i}\mathrm{V}_1 \otimes \Lambda_{\lambda_i - \sigma_i}\mathrm{V}_2 \longrightarrow \Lambda_{\lambda_i - \mu_i}(\mathrm{V}_1 \oplus \mathrm{V}_2).$$

Let $M_\gamma(L_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2))$ and $\dot{M}_\gamma(L_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2))$ be the images of the previuos modules under the Schur map $d_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2)$. The following result holds as in the classical case :

Theorem *The R-modules*

$$L_{\lambda/\mu}\mathrm{V}_1 \otimes L_{\lambda/\gamma}\mathrm{V}_2 \quad , \quad M_\gamma(L_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2))/\dot{M}_\gamma(L_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2))$$

are isomorphic. Hence the R-modules $M_\gamma(L_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2))$, $\mu \subseteq \gamma \subseteq \lambda$, *give a filtration of* $L_{\lambda/\mu}(\mathrm{V}_1 \oplus \mathrm{V}_2)$, *whose associated graded module is isomorphic to*

$$\sum_{\mu \subseteq \gamma \subseteq \lambda} L_{\gamma/\mu}\mathrm{V}_1 \otimes L_{\lambda/\gamma}\mathrm{V}_2.$$

Proof. Follow *verbatim* the proof of Theorem II. 4.11 in [A-B-W]. □

Note that the isomorphism of the theorem is in fact an isomorphism of $SE(q,\mathrm{P}^1) \otimes SE(q,\mathrm{P}^2)$-comodules.

2. The Recipe

2.1 In this Section we let R be the ring $R=\mathbb{Z}[\mathcal{Q},\mathcal{Q}^{-1}]$, $\mathcal{Q}$ an indeterminate, and take a multiplicatively antisymmetric matrix $\mathrm{P}=(p_{ij})_{i,j=1}^N$, and the YB pair $(\mathrm{V_P},\beta_{\mathcal{Q},\mathrm{P}})$, where $\mathrm{V_P}=<u_1,\ldots,u_N>$ and

$$(1)\qquad \beta_{\mathcal{Q},\mathrm{P}}(u_i\otimes u_j)=\begin{cases} u_i\otimes u_i & \text{if } i=j \\ \mathcal{Q}p_{ji}u_j\otimes u_i & \text{if } i<j \\ \mathcal{Q}p_{ji}u_j\otimes u_i+(1-\mathcal{Q}^2)u_i\otimes u_j & \text{if } i>j \end{cases}.$$

We are going to construct a filtration of $L_{\lambda/\mu}\mathrm{V_P}$ as an $SE(\mathcal{Q},\mathrm{P})$-comodule, such that the associated graded object is isomorphic to $\sum_\nu \gamma(\lambda/\mu;\nu)L_\nu\mathrm{V_P}$. As in the classical Littlewood-Richardson rule, here $\gamma(\lambda/\mu;\nu)$ stands for the number of standard tableaux of shape λ/μ filled with $\tilde{\mu}_1$ copies of 1, $\tilde{\mu}_2$ copies of 2, $\tilde{\mu}_3$ copies of 3, etc., such that the associated word (formed by listing all entries from bottom to top in each column, starting from the leftmost column) is a lattice permutation.
The construction is a suitable "deformation" of the one used in the first author's doctoral thesis, Brandeis University 1984, as illustrated for instance in [B]. We again remark that owing to Theorem (1.10) (ii), the construction holds in fact for every commutative ring R and every choice of a unit $q\in R$.

2.2 In order to embed $L_{\lambda/\mu}\mathrm{V_P}$ into a (non-skew) Schur module, let $M=\mu_1$ and consider another multiplicatively antisymmetric matrix $\mathrm{P}'=(p'_{ij})_{i,j=1}^M$, together with the YB pair $(\mathrm{V_{P'}},\beta_{\mathcal{Q},\mathrm{P}'})$, where $\mathrm{V_{P'}}=<u'_1,\ldots,u'_M>$ and $\beta_{\mathcal{Q},\mathrm{P}'}$ is defined similarly to (1) above. For convenience of notations, we shall denote $\mathrm{V_P},u_i,\mathrm{V_{P'}}$, and u'_i by V,i,V', and i', respectively.
It follows from Theorem (1.14) that the $SE(\mathcal{Q},\mathrm{P}')\otimes SE(\mathcal{Q},\mathrm{P})$-comodule $L_\lambda(\mathrm{V}'\oplus\mathrm{V})$ is isomorphic to $\sum_{\alpha\subseteq\lambda}L_\alpha\mathrm{V}'\otimes L_{\lambda/\alpha}\mathrm{V}$, up to a filtration.
Let $(L_\lambda(\mathrm{V}'\oplus\mathrm{V}))_h$ denote the sub-R-module of $L_\lambda(\mathrm{V}'\oplus\mathrm{V})$ spanned by the tableaux in which h V'-indices occur. (In this section we identify tableaux and corresponding elements of Schur modules.)
Then up to a filtration,

$$(L_\lambda(\mathrm{V}'\oplus\mathrm{V}))_h\simeq\sum_{\alpha\subseteq\lambda,|\alpha|=h}L_\alpha\mathrm{V}'\otimes L_{\lambda/\alpha}\mathrm{V},$$

as $SE(\mathcal{Q},\mathrm{P}')\otimes SE(\mathcal{Q},\mathrm{P})$-comodules.
If $(L_\lambda(\mathrm{V}'\oplus\mathrm{V}))_{\tilde{\mu}}$ denotes the sub-R-module of $L_\lambda(\mathrm{V}'\oplus\mathrm{V})$ spanned by the tableaux in which every i' occurs exactly $\tilde{\mu}_i$ times, also :

$$(2)\qquad (L_\lambda(\mathrm{V}'\oplus\mathrm{V}))_{\tilde{\mu}}\simeq\sum_{\alpha\subseteq\lambda}(L_\alpha\mathrm{V}')_{\tilde{\mu}}\otimes L_{\lambda/\alpha}\mathrm{V},$$

as $SE(\mathcal{Q},\mathrm{P})$-comodules, up to a filtration.
Since the bottom piece of the filtration relative to (2) corresponds to the (lexicographically) largest partition α, namely μ, it follows :

$$(L_\lambda\mathrm{V}')_{\tilde{\mu}}\otimes L_{\lambda/\mu}\mathrm{V}\overset{SE(\mathcal{Q},\mathrm{P})}{\hookrightarrow}(L_\lambda(\mathrm{V}'\oplus\mathrm{V}))_{\tilde{\mu}}.$$

And $rk(L_\mu \mathrm{V}')_{\tilde{\mu}} = 1$ implies that

$$L_{\lambda/\mu}\mathrm{V} \overset{SE(\mathcal{Q},\mathrm{P})}{\hookrightarrow} (L_\lambda(\mathrm{V}'\oplus\mathrm{V}))_{\tilde{\mu}},$$

as wished.
Explicitly, the embedding sends the tableau $d_{\lambda/\mu}(\mathrm{V})(a_1\otimes\cdots\otimes a_s)$, $s=l(\lambda)$, to

$$d_\lambda(\mathrm{V}'\oplus\mathrm{V})[(b^{(\mu_1)}\wedge a_1)\otimes\cdots\otimes(b^{(\mu_r)}\wedge a_r)\otimes a_{r+1}\otimes\cdots\otimes a_s],\quad r=l(\mu),$$

where we write $b^{(k)}$ for $1'\wedge 2'\wedge\cdots\wedge k'\in\Lambda_k\mathrm{V}'$. Notice that $b^{(k)}$ is a relative $SB^+(\mathcal{Q},\mathrm{P}')$-invariant.

2.3 Let $\mathbf{t}=(t_{r1},\ldots,t_{11};t_{r2},\ldots,t_{12};\ldots;t_{rs},\ldots,t_{1s})$ be a family of nonnegative integers such that

$$\sum_{i=1}^{s} t_{ji} = \mu_j \qquad \forall j=1,\ldots,r.$$

Let f denote the $SE(\mathcal{Q},\mathrm{P}')$-equivariant composite map :

$$\Lambda_{\mu_r}\mathrm{V}'\otimes\cdots\otimes\Lambda_{\mu_1}\mathrm{V}'$$

$$\downarrow {\scriptstyle \otimes_{j=r}^{1}(\Delta^{t_j}_{\Lambda\mathrm{V}'})}$$

$$(\Lambda_{t_{r1}}\mathrm{V}'\otimes\cdots\otimes\Lambda_{t_{rs}}\mathrm{V}')\otimes\cdots\otimes(\Lambda_{t_{11}}\mathrm{V}'\otimes\cdots\otimes\Lambda_{t_{1s}}\mathrm{V}')$$

$$\downarrow {\scriptstyle \varphi_{\Lambda\mathrm{V}'}(\omega_{rs})}$$

$$(\Lambda_{t_{r1}}\mathrm{V}'\otimes\Lambda_{t_{r-1,1}}\mathrm{V}'\otimes\cdots\otimes\Lambda_{t_{11}}\mathrm{V}')\otimes\cdots\otimes(\Lambda_{t_{rs}}\mathrm{V}'\otimes\Lambda_{t_{r-1,s}}\mathrm{V}'\otimes\cdots\otimes\Lambda_{t_{1s}}\mathrm{V}')$$

$$\downarrow {\scriptstyle (m^{(r)}_{\Lambda\mathrm{V}'})^{\otimes s}}$$

$$\Lambda_{t_{r1}+t_{r-1,1}+\cdots+t_{11}}\mathrm{V}'\otimes\cdots\otimes\Lambda_{t_{rs}+t_{r-1,s}+\cdots+t_{1s}}\mathrm{V}'$$

where $t_j=(t_{j1},\ldots,t_{js})$, $m^{(r)}_{\Lambda\mathrm{V}'}:\Lambda\mathrm{V}'\otimes\cdots\otimes\Lambda\mathrm{V}'\longrightarrow\Lambda\mathrm{V}'$ is obtained by iterating the multiplication, and

$$\omega_{rs}=\begin{pmatrix} 1 & 2 & 3 & \cdots & s & s+1 & s+2 & \cdots & 2s+1 & \cdots & rs \\ 1 & r+1 & 2r+1 & \cdots & (s-1)r+1 & 2 & r+2 & \cdots & 3 & \cdots & rs \end{pmatrix}$$

(cf. items (1.5) and (1.7)).
As $b^{(\mu_r)}\otimes\cdots\otimes b^{(\mu_1)}$ is a relative $SB^+(\mathcal{Q},\mathrm{P}')$-invariant, also $f(b^{(\mu_r)}\otimes\cdots\otimes b^{(\mu_1)})$ is so. We denote the latter by $b(\mathbf{t})$.

2.4 For every $\nu\subseteq\lambda$ such that $|\nu|=|\lambda|-|\mu|$, let $B(\lambda/\nu)$ denote the set of all possible $b(\mathbf{t})$ which satisfy the further equalities :

$$\sum_{j=1}^{r} t_{ji} = \lambda_i-\nu_i \qquad \forall i=1,\ldots,s.$$

For every $b \in B(\lambda/\nu)$, we call $\varphi(\nu, b)$ the restriction to $\Lambda_\nu \mathrm{V} \otimes \{b\}$ of the following composite map

$$\Lambda_\nu \mathrm{V} \otimes \Lambda_{\lambda/\nu} \mathrm{V}' \xrightarrow{\varphi_\nu(\lambda)} \Lambda_\lambda(\mathrm{V}' \oplus \mathrm{V}) \xrightarrow{d_\lambda(\mathrm{V}' \oplus \mathrm{V})} \Lambda_\lambda(\mathrm{V}' \oplus \mathrm{V}),$$

where $\varphi_\nu(\lambda)$ is obtained by tensoring the morphisms

$$\Lambda_{\nu_i} \mathrm{V} \otimes \Lambda_{\lambda_i - \nu_i} \mathrm{V}' \longrightarrow \Lambda_{\lambda_i}(\mathrm{V}' \oplus \mathrm{V}), \qquad x \otimes y \mapsto x \wedge y, \qquad i = 1, \ldots, s.$$

Proposition *The image of $\varphi(\nu, b)$ lies in $\mathrm{L}_{\lambda/\mu} \mathrm{V} \hookrightarrow \mathrm{L}_\lambda(\mathrm{V}' \oplus \mathrm{V})$.*

Proof. As $\varphi(\nu, b)$ is $SE(\mathcal{Q}, \mathrm{P}') \otimes SE(\mathcal{Q}, \mathrm{P})$-equivariant, and b is a relative $SB^+(\mathcal{Q}, \mathrm{P}')$-invariant of V'-content $\tilde{\mu}$ (i.e., it contains $\tilde{\mu}_i$ copies of i'), each element of $Im(\varphi(\nu, b))$ is a relative $SB^+(\mathcal{Q}, \mathrm{P}')$-invariant of V'-content $\tilde{\mu}$. But then we are through, thanks to Lemma (2.5) below and to the fact that $d_\mu(\mathrm{V}')(b^{(\mu_1)} \otimes \cdots \otimes b^{(\mu_r)})$ is the only canonical tableau of content $\tilde{\mu}$. □

2.5 Lemma *For every partition α, take in $L_\alpha \mathrm{V}' \otimes_{\mathrm{R}} \mathbb{Q}(\mathcal{Q})$ the element*

$$C_\alpha = d_\alpha(\mathrm{V}')(1' \wedge \cdots \wedge \alpha'_1 \otimes 1' \wedge \cdots \wedge \alpha'_2 \otimes \cdots \otimes 1' \wedge \cdots \wedge \alpha'_l), \qquad l = l(\alpha)$$

(C_α is sometimes called the "canonical tableau of $L_\alpha \mathrm{V}'$"). Then the relative $SB^+(\mathcal{Q}, \mathrm{P}')$-invariant elements of $L_\alpha \mathrm{V}' \otimes_R \mathbb{Q}(\mathcal{Q})$ are spanned (over $\mathbb{Q}(\mathcal{Q})$) by C_α.

Proof. Combine $(L_\alpha \mathrm{V}')_{\tilde{\alpha}} = R \cdot C_\alpha$ with a multiparameter version of a suitable analogue of Theorem 6.5.2 in [P-W]. □

2.6 For each $\nu \subseteq \lambda$ such that $\gamma(\lambda/\mu; \nu) \neq 0$, we wish to describe a subset of $B(\lambda/\nu)$, say $B'(\lambda/\nu)$, such that $\#\mathrm{B}'(\lambda/\nu) = \gamma(\lambda/\mu; \nu)$. Let $\mathrm{T} \in L_{\lambda/\nu} \mathrm{V}'$ be a standard tableau, of content $\tilde{\mu}$, and such that its associated word, $as(\mathrm{T}) = (a_1, \ldots, a_{|\mu|})$, is a lattice permutation. Then μ is the content of the transpose lattice permutation $(as(\mathrm{T}))\tilde{}$. (Explicitely, $(as(\mathrm{T}))\tilde{} = (\tilde{a}_1, \ldots, \tilde{a}_{|\mu|})$, where $\tilde{a}_i$ is the number of times a_i occurs in $as(\mathrm{T})$ in the range $(a_1, \ldots, a_i)$.) Let $\tilde{\mathrm{T}}$ be the tableau obtained from T by replacing every entry a_i of T by $\tilde{a}_i$. For each $i \in \{1, \ldots, s\}$ and each $j \in \{1, \ldots, r\}$, we set :

$$t_{ji} = \#\text{ of } j'\text{s occuring in the } i-\text{th row of } \tilde{\mathrm{T}}.$$

We denote by $b(\mathrm{T})$ the element $b(\mathbf{t}) \in B(\lambda/\nu)$, corresponding to this choice of t_{ji}'s.

2.7 Given any row-standard tableau T, we can consider the word $w(\mathrm{T})$ formed by writing one after the other all the rows of T, starting from the top. As all such words can be ordered lexicographically, we can say that $\mathrm{T} <_{lex} \mathrm{T}'$ if and only if $w(\mathrm{T}) <_{lex} w(\mathrm{T}')$. It is then easy to see that the following holds.

Proposition *If we write $b(\mathrm{T}) \in \Lambda_{\lambda/\nu} \mathrm{V}'$ as a linear combination of row-standard tableaux, then*

$$b(\mathrm{T}) = \pm \mathcal{Q}^* \mathrm{T} + \sum_k c_k \mathrm{T}_k, \qquad c_k \in \mathbb{Z}[\mathcal{Q}, \mathcal{Q}^{-1}],$$

where $\mathcal{Q}^*$ stands for a power of $\mathcal{Q}$, and each T_k is a row-standard tableau $<_{lex}$ T. $\square$

Since there are exactly $\gamma(\lambda/\mu;\nu)$ tableaux $\mathrm{T} \in L_{\lambda/\nu}\mathrm{V}'$ which are standard, of content $\tilde{\mu}$, and such that $as(\mathrm{T})$ is a lattice permutation, the above Proposition implies that the elements $b(\mathrm{T})$ form a subset of $B(\lambda/\nu)$ of cardinality $\gamma(\lambda/\mu;\nu)$. It is precisely this subset which we call $B'(\lambda/\nu)$.

2.8 Consider the family of elements of $L_{\lambda/\mu}\mathrm{V} \hookrightarrow L_\lambda(\mathrm{V}' \oplus \mathrm{V})$:

$$\mathcal{F} = \{\varphi(\nu, b)(x) |\ \gamma(\lambda/\mu;\nu) \neq 0,\ b \in B'(\lambda/\nu),\ \text{and } d_\nu(\mathrm{V})(x) \text{ is a standard tableau}\}.$$

We claim that $\mathcal{F}$ is an R-basis of $L_{\lambda/\mu}\mathrm{V}$.

Proposition *The elements of $\mathcal{F}$ are linearly indipendent over R.*

Proof. Suppose that there exist nonzero coefficients $r_{\nu,b,x} \in R$ such that $\sum_{\mathcal{F}} r_{\nu,b,x}\varphi(\nu,b)(x) = 0$, i.e., such that $\sum_{\nu,b,x} r_{\nu,b,x} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_\nu(\lambda)(x \otimes b)) = 0$ in $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$. This is the same as

$$(3) \qquad \sum_{\nu,b} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_\nu(\lambda)(y_{\nu,b} \otimes b)) = 0,$$

where $y_{\nu,b} = \sum_x r_{\nu,b,x} x$. Let ν_0 be the (lexicographically) smallest ν occuring in (3). Order the set $B'(\lambda/\nu_0) = \{b(\mathrm{T}_1), \ldots, b(\mathrm{T}_p)\}$ as follows :

$$b(\mathrm{T}_i) < b(\mathrm{T}_j) \text{ if and only if } w(\mathrm{T}_i) <_{lex} w(\mathrm{T}_j).$$

Let $b_0 = b(\mathrm{T}_0)$ be the highest $b(\mathrm{T}_i) \in B'(\lambda/\nu_0)$ occuring in $\sum_{\nu,b} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_\nu(\lambda)(y_{\nu,b} \otimes b))$. Clearly, $d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes b_0))$ is not in general a linear combination of standard tableaux of $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$, with respect to the order $1 < \cdots < N < 1' < \cdots < M'$, since violations of column-standardness may occur in b_0. Apply therefore to $d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes b_0))$ the straightening law of $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$ with respect to $1 < \cdots < N < 1' < \cdots < M'$. One gets (recall Proposition (2.7)) :

$\pm\mathcal{Q}^* d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes \mathrm{T}_0))+$ (a linear combination of standard tableaux with V-shape $> \nu_0$) +(a linear combination of standard tableaux with V-shape $= \nu_0$ and V$'$-part $<_{lex}$ T_0).

Because of our choice of ν_0 and b_0, (3) then implies that $d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(y_{\nu_0,b_0} \otimes \mathrm{T}_0)) = 0$, i.e.,

$$\sum_x r_{\nu_0,b_0,x} d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(x \otimes \mathrm{T}_0)) = 0.$$

But this is a linear combination of standard tableaux in $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$, with respect to the order $1 < \cdots < N < 1' < \cdots < M'$, so that $r_{\nu_0,b_0,x} = 0$ for each x, which contradicts our assumption on the coefficients $r_{\nu,b,x}$. $\square$

2.9 Corollary *$\mathcal{F}$ is a basis for $L_{\lambda/\mu}\mathrm{V} \otimes_R \mathbb{Q}(\mathcal{Q})$.*

Proof. By definition of $\mathcal{F}$, $\#\mathcal{F} = rk(L_{\lambda/\mu}\mathrm{V})$. By Theorem (1.10)(ii), the latter rank is constant on all rings. So proposition (2.8) says that $\mathcal{F}$ is a basis for the vector space $L_{\lambda/\mu}\mathrm{V}\otimes_R \mathbb{Q}(\mathcal{Q})$. $\square$

2.10 Corollary *$\mathcal{F}$ is a basis for $L_{\lambda/\mu}\mathrm{V}$.*

Proof. It suffices to show that $\mathcal{F}$ is a system of generators for $L_{\lambda/\mu}\mathrm{V}$. Let $y \in L_\lambda(\mathrm{V}' \oplus \mathrm{V})$ be any tableau of type

$$\begin{array}{llllllll}
1' & \cdot & \cdot & \cdot & \mu'_1 & \circ & \circ & \circ \\
1' & \cdot & \cdot & \mu'_2 & \circ & \circ & \circ & \\
1' & \cdot & \mu'_3 & \circ & \circ & & & \\
\cdot & \cdot & \circ & \circ & & & & \\
\cdot & \cdot & \circ & \circ & & & & \\
\circ & \circ & \circ & & & & & \\
\circ & \circ & & & & & &
\end{array},$$

where the little circles stand for basis elements of V.
Since $y \in L_{\lambda/\mu}\mathrm{V}$, Corollary (2.9) says that in the quotient field of R, there exist (unique) coefficients $q_{\nu,b,x}$, such that

$$y = \sum_{\mathcal{F}} q_{\nu,b,x}\varphi(\nu, b)(x). \tag{4}$$

To both sides of (4), apply the straightening law with respect to $1 < \cdots < N < 1' < \cdots < M'$. In the left-hand side, only coefficients in R occur. In the right-hand side, if ν_0 denotes the smallest V-shape coupled with a nonzero $\sum_x q_{\nu,b,x}x$, and $b_0 = b(\mathrm{T}_0)$ denotes the highest element of $B'(\lambda/\nu_0)$ (cf. ordering in the proof of Proposition (2.8)) occurring with a nonzero $\sum_x q_{\nu_0,b,x}x$, we find that the term $\pm\mathcal{Q}^* d_\lambda(\mathrm{V}' \oplus \mathrm{V})(\varphi_{\nu_0}(\lambda)(\sum_x q_{\nu_0,b_0,x}x\otimes\mathrm{T}_0))$ must cancel with something in the left-hand side; since each $d_{\nu_0}(\mathrm{V})(x) \in L_\nu\mathrm{V}$ is standard, it follows that $q_{\nu_0,b_0,x} \in R$ for every x.
Write next (4) as :

$$y - \sum_x q_{\nu_0,b_0,x}\varphi(\nu_0, b_0)(x) = \sum_{(\nu,b)\neq(\nu_0,b_0)} \varphi(\nu, b)(\sum_x q_{\nu,b,x}x). \tag{4'}$$

Reasoning for (4′) as done for (4), it follows that $q_{\nu_1,b_1,x} \in R$, where (ν_1, b_1) is the pair (ν, b) coming immediately before (ν_0, b_0) in the total ordering :
$(\nu, b) < (\nu', b')$ if and only if either $\nu > \nu'$, or $\nu = \nu'$ and $b < b'$ in the ordering of $B'(\lambda/\nu)$ given in the proof of Proposition (2.8).
Repeating the argument as many times as necessary, the proof is completed. $\square$

2.11 Theorem *Up to a filtration, $L_{\lambda/\mu}\mathrm{V} \simeq \sum_\nu \gamma(\lambda/\mu;\nu)L_\nu\mathrm{V}$ as $SE(\mathcal{Q},\mathrm{P})$-comodules.*

Proof. For every ν such that $\gamma(\lambda/\mu;\nu) \neq 0$, let M_ν denote the R-span (in $L_\lambda(\mathrm{V}' \oplus \mathrm{V})$) of all elements $\varphi(\tau, b)(x)$ of $\mathcal{F}$ such that $\tau \geq \nu$. Also let $\dot{M}_\nu$ denote the R-span of all $\varphi(\tau, b)(x)$ such that $\tau > \nu$. Clearly, we have the isomorphism of free R-modules :

$$M_\nu/\dot{M}_\nu \xrightarrow{\psi_\nu} L_\nu\mathrm{V} \oplus \cdots \oplus L_\nu\mathrm{V} \quad (\gamma(\lambda/\mu;\nu) \text{ summands}).$$

$\{M_\nu\}$ will be the required filtration, if we show that each ψ_ν is an $SE(\mathcal{Q},\mathrm{P})$-isomorphism. In order to do so, it suffices to prove that for every fixed $b_0 \in B'(\lambda/\nu)$, and for every basis element $y \in \Lambda_\nu \mathrm{V}$, $\varphi(\nu,b_0)(y) - \varphi(\nu,b_0)(\sum r_i x_i) \in \dot{M}_\nu$, where $\sum r_i d_\nu(\mathrm{V})(x_i)$ is obtained by application to the tableau $d_\nu(\mathrm{V})(y)$ of the straightening law of $L_\nu\mathrm{V}$. Notice however that $\varphi(\nu,b_0)(y) \in L_{\lambda/\mu}\mathrm{V} \subseteq L_\lambda(\mathrm{V}' \oplus \mathrm{V})$ can be written in two ways :

$$\varphi(\nu,b_0)(y) = \sum_{\mathcal{F}} r_{\tau,b,x}\varphi(\tau,b)(x), \tag{5}$$

by Corollary (2.10), and

$$\varphi(\nu,b_0)(y) = \sum r_i\varphi(\nu,b_0)(x_i) + L.C., \tag{6}$$

where $L.C.$ denotes a linear combination of tableaux, standard with respect to $1 < \cdots < N < 1' < \cdots < M'$, and with V-part $> \nu$. This last equality is obtained by eliminating in the V-part of $\varphi(\nu,b_0)(y)$ all violations of standardness, with respect to $1 < \cdots < N < 1' < \cdots < M'$.
Comparing (5) and (6), it follows that

$$\varphi(\nu,b_0)(y) - \varphi(\nu,b_0)(\sum r_i x_i) = \sum_{\mathcal{F}} r_{\tau,b,x}\varphi(\tau,b)(x)$$

with $r_{\tau,b,x} = 0$ whenever $\tau \leq \nu$. Hence $\varphi(\nu,b_0)(y) - \varphi(\nu,b_0)(\sum r_i x_i) \in \dot{M}_\nu$ as wished. $\square$

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